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Seven-dimensional Einstein manifolds from Tod–Hitchin geometry

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Abstract

We construct infinitely many seven-dimensional Einstein metrics of weak holonomy G_2. These metrics are defined on principal SO(3) bundles over four-dimensional Bianchi IX orbifolds with the Tod–Hitchin metrics. The Tod–Hitchin metric has an orbifold singularity parametrized by an integer, and is shown to be similar near the singularity to the Taub-NUT de Sitter metric with a special charge. We show, however, that the seven-dimensional metrics on the total space are actually smooth. The geodesics on the weak G_2 manifolds are discussed. It is shown that the geodesic equation is equivalent to the Hamiltonian equation of an interacting rigid body system. We also discuss M-theory on the product space of AdS_4 and the seven-dimensional manifolds, and the dual gauge theories in three dimensions.

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1. Introduction

M-theory compactifications on special holonomy manifolds have attracted much attention, because they preserve some supersymmetry and allow one to examine dynamical aspects of a large class of supersymmetric gauge theories [1]. For example, it is known that there are eight-dimensional Ricci flat manifolds with holonomy Sp(2), SU(4) and Spin(7) except for the trivial one, and M-theory compactifications on them correspond to three-dimensional gauge theories with $\mathcal{N} = 3$, 2 and 1 supersymmetry, respectively. For a non-compact eight-dimensional special holonomy manifold, M-theory on it is interpreted as a worldvolume theory on an M2-brane with a special holonomy manifold as the transverse space. This is closely related to the supersymmetric M-theory solution $\mathrm{AdS}_4 \times M$ with compact seven-dimensional Einstein manifold M. For weak G_2 manifolds M, namely, 3-Sasakian, Sasaki–Einstein and proper weak G_2 manifolds, the M-theory solutions $\mathrm{AdS}_4 \times M$ are AdS/CFT dual to $\mathcal{N} = 3$, 2 and 1 superconformal field theories on the boundary of AdS_4 [2–5]. The brane solution naturally interpolates between $\mathrm{AdS}_4 \times M$ in the near horizon limit and $\mathbb{R}^{1,2} \times C(M)$, where $C(M)$ is the cone over M with the special holonomy SP(2), SU(4) or Spin(7), and the gauge theories on both sides are related by the RG flow [6].

In this paper, we construct infinitely many seven-dimensional Einstein metrics admitting 3-Sasakian and proper weak G_2 structures.[1] These metrics are defined on compact manifolds M_k parametrized by an integer $k \geq 3$: principal SO(3) bundles over four-dimensional Bianchi IX orbifolds with the Tod–Hitchin metrics [9–11]. The Tod–Hitchin metric has an orbifold singularity parametrized by the integer k. However, the singularity is resolved by adding the fiber SO(3), and so the total spaces M_k become smooth manifolds. Our compact manifolds contain manifolds S^7, $N^{0,1,0}$ and the squashed S^7 as special homogeneous cases for $k = 3, 4$ [12]. For generic k, the metrics on M_k are inhomogeneous and admit $\mathrm{SO}(3) \times \mathrm{SO}(3)$ isometry. This implies that the dual gauge theories in three dimensions are $\mathcal{N} = 3$ supersymmetric with SO(3) flavor for 3-Sasakian manifolds M_k, and $\mathcal{N} = 1$ supersymmetric with $\mathrm{SO}(3) \times \mathrm{SO}(3)$ flavor for proper weak G_2 manifolds M_k. We examine the geodesics on M_k using a Hamiltonian formulation on the cotangent bundle T^*M_k. The geodesic equation is equivalent to the Hamiltonian equation of an interacting rigid body system. We find some special solutions, which may be useful for considering the Penrose limit of our metrics.

In the draft [13],[2] Grove, Wilking and Ziller proved that 3-Sasakian orbifolds M_k corresponding to the Tod–Hitchin orbifolds are manifolds with the following properties: (a) for odd k, they have the same cohomology ring as an S^3-bundle over S^4, (b) for even k, they have the same cohomology ring as a general Aloff–Wallach space, (c) in both cases, it carries an invariant cohomogeneity 1 structure by $S^3 \times S^3$. In addition, the proper weak G_2 orbifolds M_k can also be made smooth by the method of K. Galicki and S. Salamon [14].[3] Our study provides a concrete procedure for resolving orbifold singularities which is familiar to physicists, and the explicit forms of the 3-Sasakian and proper weak G_2 metrics.

This paper is organized as follows. In Section 2, we introduce the Tod–Hitchin geometry, and explain the relation to the Atiyah–Hitchin manifold [15]. We show that the Tod–Hitchin geometry is well approximated by the Taub-NUT de Sitter geometry with a special charge. In Section 3, we

[1] Recently, infinitely many Sasaki–Einstein metrics were constructed in [7,8]

[2] After submitting this paper to the e-print archives, we received from K. Galicki the draft of a talk given by W. Ziller, which is referred to in [8]. We thank K. Galicki for this.

[3] We thank K. Galicki for pointing this out to us.

construct infinitely many seven-dimensional Einstein metrics of weak holonomy G_2 on compact manifolds. We also discuss the geodesics on the weak G_2 manifolds, in Section 4. In the last section, we comment on the M-theory solutions $\mathrm{AdS}_4 \times M_k$ and the dual gauge theories in three dimensions. In Appendix A, we present the anti-self-dual condition for the Bianchi IX Einstein metric. We summarize the relation between the Tod–Hitchin metric and the Painlevé VI solution in Appendix B. In Appendix C, the G_2 structure of the metric is given.

2. ASD Einstein metrics on the four-dimensional Bianchi IX manifold

In this section, we consider Bianchi IX Einstein metrics with positive cosmological constant. By using the SO(3) left-invariant 1-forms σ_i $(i = 1, 2, 3)$, the metric can be written in the form

$$g = \mathrm{d}t^2 + a^2(t)\sigma_1^2 + b^2(t)\sigma_2^2 + c^2(t)\sigma_3^2. \tag{2.1}$$

In the biaxial case, the general solution to the Einstein equation $\mathrm{Ric}(g) = \Lambda g$ has three parameters, the mass m, the Nut charge n and the cosmological constant Λ;

$$g_{\{m,n,\Lambda\}} = \frac{r^2 - n^2}{\Delta(r)}\mathrm{d}r^2 + \frac{4n^2\Delta(r)}{r^2 - n^2}\sigma_1^2 + (r^2 - n^2)(\sigma_2^2 + \sigma_3^2), \tag{2.2}$$

where

$$\Delta(r) = r^2 - 2mr + n^2 + \Lambda\left(n^4 + 2n^2r^2 - \frac{1}{3}r^4\right). \tag{2.3}$$

The anti-self-dual (ASD) condition for the Weyl curvature determines m in terms of n and Λ as

$$m = -n\left(1 + \frac{4}{3}\Lambda n^2\right), \tag{2.4}$$

in which case

$$\Delta(r) = \frac{\Lambda}{3}(r + n)^2(r_+ - r)(r - r_-), \qquad r_\pm = n \pm \sqrt{4n^2 + \frac{3}{\Lambda}}. \tag{2.5}$$

Then the metric (2.2) becomes the ASD Taub-NUT de Sitter metric [16,17] given by

$$g_{\{n,\Lambda\}} = \frac{\mathrm{d}r^2}{F(r)} + 4n^2F(r)\sigma_1^2 + (r^2 - n^2)(\sigma_2^2 + \sigma_3^2), \tag{2.6}$$

where

$$F(r) = \frac{\Lambda}{3}\left(\frac{r + n}{r - n}\right)(r_+ - r)(r - r_-). \tag{2.7}$$

For $\Lambda = 0$, the metric reduces to the ASD Taub-NUT metric [18],

$$g_{\{n,0\}} = \left(\frac{r - n}{r + n}\right)\mathrm{d}r^2 + 4n^2\left(\frac{r + n}{r - n}\right)\sigma_1^2 + (r^2 - n^2)(\sigma_2^2 + \sigma_3^2). \tag{2.8}$$

We shall now restrict our attention to the metric (2.6) with the special Nut charge

$$n = \sqrt{\frac{3}{\Lambda(k^2 - 4)}}, \tag{2.9}$$

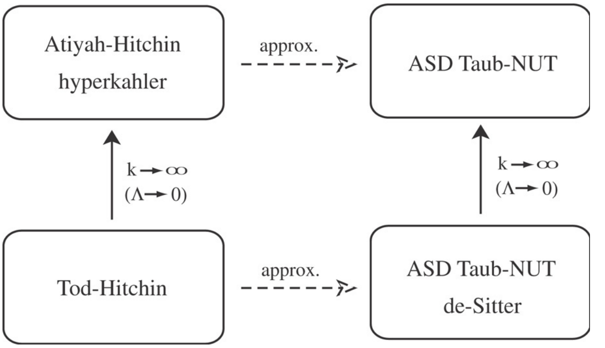

Fig. 1. The relation among metrics.

which is a family of ASD Einstein metrics $g_k \equiv g_{\{n=\sqrt{3/\Lambda(k^2-4)},\ \Lambda\}}$ parametrized by the integer $k \geq 3$. Each metric g_k has the following properties (see Fig. 1):

(a) When the coordinate r is taken to lie in the interval $n \leq r \leq r_+$, the metric has singularities at the boundaries; there is an orbifold singularity at $r = r_+$, while there is a curvature singularity at another boundary $r = n$.
(b) The metric gives an approximation to the Tod–Hitchin metric.
(c) As $k \to \infty$ and $\Lambda \to 0$ keeping $\Lambda k^2 = 3$, the metric converges to the ASD Taub-NUT metric (2.8) with a negative mass parameter ($n = 1$) which gives the asymptotic form of the Atiyah–Hitchin hyperkähler metric.

In the following, we will explain these points in some detail. For this purpose, we start with an explanation of some relevant aspects of the Tod–Hitchin metrics. Tod and Hitchin constructed a family of ASD Einstein metrics (Tod–Hitchin metrics) on the Bianchi IX orbifold, parametrized by an integer $k \geq 3$ [9–11]. These solutions are written in the triaxial form and have a compactification as metrics with orbifold singularities. These may be thought of as a resolution of the curvature singularity in the ASD Taub-NUT de Sitter metric g_k. Each Tod–Hitchin metric g_k^{TH} is given by a solution to the Painlevé VI equation (see Appendix B). For lower k the metric takes the form [11,16]:

- $k = 3$

$$g_3^{\mathrm{TH}} = \mathrm{d}t^2 + 4\sin^2 t\sigma_1^2 + 4\sin^2\left(\frac{2}{3}\pi - t\right)\sigma_2^2 + 4\sin^2\left(t + \frac{2}{3}\pi\right)\sigma_3^2, \tag{2.10}$$

which gives the standard metric on S^4 written in the triaxial form;
- $k = 4$

$$g_4^{\mathrm{TH}} = \mathrm{d}t^2 + \sin^2 t\sigma_1^2 + \cos^2 t\sigma_2^2 + \cos^2 2t\sigma_3^2, \tag{2.11}$$

which gives the Fubini–Study metric on $\mathbb{CP}^2$.
- $k = 6,\ 8$

The metric can be written as

$$g_k^{\mathrm{TH}} = h(r)\mathrm{d}r^2 + a^2(r)\sigma_1^2 + b^2(r)\sigma_2^2 + c^2(r)\sigma_3^2, \tag{2.12}$$

where the components are given for $k = 6$ by

$$h^2 = \frac{3(1+r+r^2)}{r\,(r+2)^2\,(2r+1)^2}, \qquad a^2 = \frac{3(1+r+r^2)}{(r+2)\,(2r+1)^2},$$
$$b^2 = \frac{3r\,(1+r+r^2)}{(r+2)^2\,(2r+1)}, \qquad c^2 = \frac{3(r^2-1)^2}{(1+r+r^2)\,(r+2)\,(2r+1)}, \tag{2.13}$$

and for $k = 8$ by

$$h^2 = \frac{4(1+r)(3-2r+r^2)(1-2r+3r^2)(1+2r+3r^2)}{(1-r)\,r\,(1+r^2)(1+2r-r^2)^2\,(3+2r+r^2)^2},$$
$$a^2 = \frac{4(1-r)(1+r)^3\,(3-2r+r^2)(1-2r+3r^2)}{(1+2r-r^2)(3+2r+r^2)^2\,(1+2r+3r^2)},$$
$$b^2 = \frac{16r\,(1-2r+3r^2)(1+2r+3r^2)}{(1+2r-r^2)(3-2r+r^2)(3+2r+r^2)^2},$$
$$c^2 = \frac{4\,(1+r^2)(3-2r+r^2)(1-2r-r^2)^2(1+2r+3r^2)}{(1+2r-r^2)^2(3+2r+r^2)^2(1-2r+3r^2)}. \tag{2.14}$$

Among the Tod–Hitchin metrics, those with $k = 3$ and 4 are exceptional, i.e. there is no singularity. The solutions with higher k are determined by the non-trivial solutions to the Painlevé equation, and in the limit $k \to \infty$ together with a suitable scaling of Λ the solution approaches the Atiyah–Hitchin metric. In the paper [11], Hitchin found a systematic algebraic way of finding solutions of the Painlevé equation. However, it is not easy to write down these solutions explicitly. To examine such a solution, we consider the local metric near the boundary by using expansions of the solution (2.1) to the Einstein equation.

To begin with, we discuss boundary conditions. Let us impose a compact condition for the Bianchi IX manifold $\simeq I \times \mathrm{SO}(3)$, where I is the closed interval $[t_1, t_2] \subset \mathbb{R}$. Furthermore we require that singularities at the boundaries, t_1 and t_2, are described by Bolts or Nuts so that there are three types, Nut–Nut, Bolt–Nut and Bolt–Bolt. The Tod–Hitchin metric belongs to Bolt–Bolt type: near $t = t_1$, the metric is written as

$$g_k^{\mathrm{TH}} \sim \mathrm{d}t^2 + \frac{4t^2}{(k-2)^2}\,\sigma_1^2 + L^2\,(\sigma_2^2 + \sigma_3^2). \tag{2.15}$$

On the other hand, near $t = t_2$

$$g_k^{\mathrm{TH}} \sim \mathrm{d}t^2 + M^2\,(\sigma_1^2 + \sigma_2^2) + 4t^2\,\sigma_3^2. \tag{2.16}$$

It should be noticed that at one side of the boundaries the coefficient of σ_1 vanishes, while at the other side it is the coefficient of σ_3 that vanishes. The constant L in (2.15) is fixed by the ASD condition as

$$L^2 = \frac{3}{\Lambda}\frac{k}{k-2}. \tag{2.17}$$

We introduce Euler angles (θ, ϕ, ψ) of SO(3) with the ranges

$$0 \le \theta < \pi, \quad 0 \le \phi < 2\pi, \quad 0 \le \psi < 2\pi, \tag{2.18}$$

so that

$$\sigma_1 = \mathrm{d}\psi + \cos\theta\,\mathrm{d}\phi, \tag{2.19}$$

$$\sigma_2 = \cos\psi \mathrm{d}\theta + \sin\psi \sin\theta \mathrm{d}\phi, \tag{2.20}$$

$$\sigma_3 = -\sin\psi \mathrm{d}\theta + \cos\psi \sin\theta \mathrm{d}\phi. \tag{2.21}$$

From (2.16) the Tod–Hitchin metric behaves near $t = t_1$, with fixed θ and ϕ, as

$$g_k^{\mathrm{TH}} \sim \mathrm{d}t^2 + \frac{t^2}{(k-2)^2}\mathrm{d}(2\psi)^2. \tag{2.22}$$

Therefore, the metric has an orbifold singularity with angle $2\pi/(k-2)$ around $\mathbb{RP}^2$ when we impose the $\mathbb{Z}_2$ identification $\psi \equiv \psi + \pi$. Similar arguments require further $\mathbb{Z}_2$ identification at the other endpoint $t = t_2$, and then the metric extends smoothly over $\mathbb{RP}^2$ at $t = t_2$. Thus, the principal orbits are $SO(3)/\mathbb{Z}_2 \times \mathbb{Z}_2$ and the Tod–Hitchin metrics are defined on $\mathbb{RP}^2 \cup [(t_1, t_2) \times SO(3)/(\mathbb{Z}_2 \times \mathbb{Z}_2)] \cup \mathbb{RP}^2$, which is topologically equivalent to S^4 [11,16]. The Taub-NUT de Sitter metric g_k near the boundary $r = r_+$ coincides with the asymptotic form (2.15), on setting $t = \int_r^{r_+}(1/\sqrt{F(r)})\mathrm{d}r$. However, the metric on the other boundary $r = n$ is different from (2.16), and turns out to have the curvature singularity. The higher order expansions with the initial conditions (2.15) and (2.16) reveal the further structure of the Tod–Hitchin metric.

Using the Einstein equation (see Appendix A), we find the following asymptotic behavior of the Tod–Hitchin metric in the form (2.1) near the boundary:

(1) Near $t = t_1$

$$\begin{aligned} a(t) &\sim \frac{2t}{k-2} + \sum_{j=1}^{\infty} a_{2j+1}t^{2j+1}, \\ b(t) &\sim L + \sum_{j=1}^{\infty} b_{2j}t^{2j} + \delta\, t^{k-2}\left(1 + \sum_{n=1}^{\infty}\delta_n t^n\right), \\ c(t) &\sim L + \sum_{j=1}^{\infty} b_{2j}t^{2j} + \delta\, t^{k-2}\left(-1 + \sum_{n=1}^{\infty}\hat{\delta}_n t^n\right). \end{aligned} \tag{2.23}$$

Here the expansion includes one free parameter δ, and the remaining coefficients are determined by k, δ and L (see (2.17)). In this expansion, the terms multiplied by δ represent the deviation from the biaxial form. It should be noticed that the deviation is "small" because of the presence of the suppression factor t^{k-2}.[4]

(2) Near $t = t_2$

$$\begin{aligned} a(t) &\sim M + a_1 t + \sum_{j=2}^{\infty} a_j t^j, \\ b(t) &\sim M - a_1 t + \sum_{j=2}^{\infty} b_j t^j. \\ c(t) &\sim 2t + \sum_{j=1}^{\infty} c_{2j+1}t^{2j+1}. \end{aligned} \tag{2.24}$$

[4] In [19], it was shown that there exists a similar expansion to (2.23) for a certain class of higher dimensional Einstein metrics.

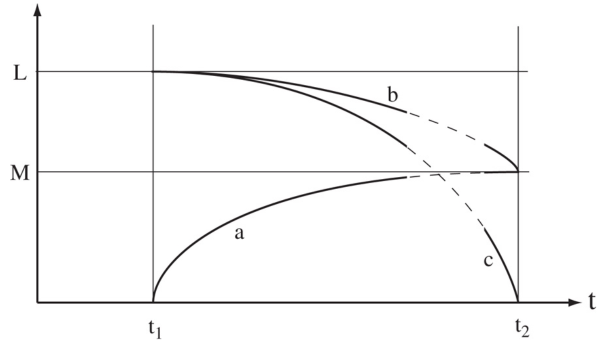

Fig. 2. An illustration of the Tod–Hitchin metric.

Here the expansion includes one free parameter M, and the ASD condition requires

$$a_1^2 = \frac{1}{4} + \frac{M^2 \Lambda}{12}. \tag{2.25}$$

The remaining coefficients are successively determined.

The Tod–Hitchin metric corresponds to that with a certain value δ in (2.23) or M in (2.24); the determination of these values requires the global information connecting the local solutions near the boundaries, which is lacking in our analysis (see Fig. 2). In particular, for the exact solutions (2.10)–(2.14), the parameters (δ, M, Λ) are given by

(a) $k = 3$: $(1,\ \sqrt{3},\ 3)$, $0 \leq t \leq \pi/3$.
(b) $k = 4$: $(3/4,\ 1/\sqrt{2},\ 6)$, $0 \leq t \leq \pi/4$.
(c) $k = 6$: $(5\sqrt{6}/72,\ 1/\sqrt{3},\ 3)$, $0 \leq r \leq \infty$.
(d) $k = 8$: $(63\sqrt{3}/2048,\ \sqrt{3-2\sqrt{2}},\ 3)$, $\sqrt{2}-1 \leq r \leq 1$.

When we consider the case with large k, the expansion (2.23) implies that the biaxial solutions approximate well the Tod–Hitchin metrics near the boundary $t = t_1$. We find that the ASD Taub-NUT de Sitter solution g_k exactly reproduces the expansion (2.23) with $\delta = 0$. In the limit $k \to \infty$, Eq. (2.23) yields $b(t) \sim c(t)$, which is consistent with the asymptotic behavior of the Atiyah–Hitchin metric. Indeed, the Atiyah–Hitchin metric behaves like the ASD Taub-NUT metric with exponentially small corrections [20].

The Atiyah–Hitchin manifold is identified as the moduli space of the three-dimensional $\mathcal{N} = 4$ SU(2) gauge theory [21,22]. The vacuum expectation values of bosonic fields of the theory, three SO(3) scalars ϕ_i and one scalar σ dual of the photon, parametrize the Atiyah–Hitchin manifold. The hyperkähler structure of the Atiyah–Hitchin manifold ensures the $\mathcal{N} = 4$ supersymmetry. In the region of large $\langle \phi_i \rangle$, the monopole correction is suppressed and the moduli are well approximated by the Taub-NUT geometry with a negative charge. On the other hand, near the origin, the Tod–Hitchin geometry provides a good approximation even if k is small, and thus one can expect the gauge theory near the origin of the moduli to be well described by that with the Tod–Hitchin geometry as the moduli. In this approximation, the metric on the moduli becomes simpler but the gauge theory fails to be supersymmetric. This is because the Tod–Hitchin geometry is not Kähler, while the Atiyah–Hitchin manifold is hyperkähler. As we have seen, the Tod–Hitchin geometry converges to the Atiyah–Hitchin manifold in the limit $k \to \infty$ together with $\Lambda \to 0$. It is interesting to consider the gauge theory with the Tod–Hitchin geometry as the moduli and to reveal the role of the limit. In this limit, the supersymmetry recovers and the moduli becomes non-compact on sending the orbifold singularity of the Tod–Hitchin geometry to infinity. On the other hand, to study the region near the orbifold singularity, it will be useful to examine the theory with the Taub-NUT de Sitter geometry as the moduli. This is left for future investigations.

3. Einstein metrics on compact weak G_2 manifolds

In this section we shall describe seven-dimensional geometries based on ASD Bianchi IX orbifolds $\mathcal{O}_k$ with the Tod–Hitchin metrics g_k^{TH}. As discussed in the previous section, the Tod–Hitchin metric is defined on S^4 with an orbifold singularity parametrized by the integer k. However, we shall show that a principal SO(3) bundle $M_k \to \mathcal{O}_k$ is actually smooth and the total space M_k admits Einstein metrics of weak holonomy G_2. In this way, we obtain an infinite series of seven-dimensional compact Einstein manifolds.

Let ϕ be an SO(3)-connection on M_k; it is locally written as

$$\phi = s^{-1}As + s^{-1}\mathrm{d}s, \quad s \in \mathrm{SO}(3). \tag{3.26}$$

Here, A is an SO(3)-valued local 1-form on $\mathcal{O}_k$ and $s^{-1}\mathrm{d}s$ is regarded as the Maurer–Cartan form. We let ϕ^i denote the component of the connection with respect to the standard basis $\{E^i\}$ of SO(3) which satisfies the Lie bracket relation $[E^i, E^j] = \epsilon_{ijk}E^k$. The left-invariant 1-forms $\tilde{\sigma}_i$ are defined by $s^{-1}\mathrm{d}s = \tilde{\sigma}_i E^i$ and so the Eq. (3.26) may be written as $\phi^i = s_{ji}A^j + \tilde{\sigma}_i$ by using the adjoint representation $s^{-1}E^i s = s_{ij}E^j$. Given a metric $\alpha = (\alpha_{ij})$ on SO(3), then the Kaluza–Klein metric on M_k takes the form

$$\boldsymbol{g}_k = \alpha_{ij}\phi^i\phi^j + g_k^{\mathrm{TH}}. \tag{3.27}$$

The Einstein equation can be solved by imposing the following conditions:

(1) A^i is an SO(3) Yang–Mills instanton on $\mathcal{O}_k$.
(2) The metric α has a diagonal form; $\alpha = \mathrm{diag}(\alpha_1^2, \alpha_2^2, \alpha_3^2)$ where α_i are constants.

The instanton is given by the self-dual spin connection, $A^i = -\omega_{0i} - \frac{1}{2}\epsilon_{ijk}\omega_{jk}$. Using the explicit formula (A.4), it is written as $A^i = K_i\sigma_i$ with

$$\begin{aligned} K_1 &= \dot{a} + \frac{-a^2+b^2+c^2}{2bc}, \\ K_2 &= \dot{b} + \frac{a^2-b^2+c^2}{2ac}, \\ K_3 &= \dot{c} + \frac{a^2+b^2-c^2}{2ab}. \end{aligned} \tag{3.28}$$

Thus, the seven-dimensional Einstein equations with cosmological constant λ are equivalent to

$$\frac{\alpha_1^4 - (\alpha_2^2-\alpha_3^2)^2}{2\alpha_1^2\alpha_2^2\alpha_3^2} + \left(\frac{\Lambda}{3}\right)^2\alpha_1^2 = \lambda, \quad \Lambda - \frac{1}{2}\left(\frac{\Lambda}{3}\right)^2(\alpha_1^2+\alpha_2^2+\alpha_3^2) = \lambda, \tag{3.29}$$

and the two equations with cyclic permutation of $\alpha_1, \alpha_2, \alpha_3$. These can be solved easily, and one has two solutions,

$$\alpha = \beta_\ell\,\mathrm{diag}(1,1,1), \quad \beta_\ell = \frac{3}{\ell\Lambda} \tag{3.30}$$

with $\lambda = \Lambda\frac{2\ell-1}{2\ell}$ ($\ell = 1$ or 5). Using the right-invariant 1-forms $\hat{\sigma}_i$ ($s\mathrm{d}s^{-1} = \hat{\sigma}_i E^i$) and the Tod–Hitchin metric in the form (2.1), we find two types of seven-dimensional Einstein metrics:

$$\boldsymbol{g}_k^{(\ell)} = \mathrm{d}t^2 + a^2(t)\sigma_1^2 + b^2(t)\sigma_2^2 + c^2(t)\sigma_3^2 + \beta_\ell(K_i(t)\sigma_i - \hat{\sigma}_i)^2. \tag{3.31}$$

The conditions (1) and (2) also induce a G_2-structure on M_k as follows: Recall that the G_2-structure is characterized by a global 1-form ω, which is written locally as

$$\begin{aligned}\omega &= \theta^1\wedge\theta^2\wedge\theta^3 + \theta^1\wedge(\theta^4\wedge\theta^5 + \theta^6\wedge\theta^7)\\ &\quad + \theta^2\wedge(\theta^4\wedge\theta^6 + \theta^7\wedge\theta^5) + \theta^3\wedge(\theta^4\wedge\theta^7 + \theta^5\wedge\theta^6),\end{aligned} \tag{3.32}$$

where $\{\theta^\alpha; \alpha = 1, 2, \ldots, 7\}$ is a fixed orthonormal basis of the seven-dimensional metric $\boldsymbol{g}_{\mathrm{diag}}$ (see Appendix C). The condition of weak holonomy G_2 is defined by $\mathrm{d}\omega = c * \omega$ where $*$ is the Hodge star operation associated with $\boldsymbol{g}_{\mathrm{diag}}$ and c is a constant. Under (1) and (2), the weak G_2 condition reproduces the metric (3.31). The holonomy group $\mathrm{Hol}(\bar{\boldsymbol{g}}_k^{(\ell)})$ of the metric cone $(C(M_k), \bar{\boldsymbol{g}}_k^{(\ell)}) = (R_+ \times M_k, \mathrm{d}\tau^2 + \tau^2\boldsymbol{g}_k^{(\ell)})$ is contained in Spin(7) [23,14]:

(A) $\mathrm{Hol}(\bar{\boldsymbol{g}}_k^{(1)}) = \mathrm{Sp}(2) \subset \mathrm{Spin}(7)$ and $(M_k, \boldsymbol{g}_k^{(1)})$ is a 3-Sasakian manifold.
(B) $\mathrm{Hol}(\bar{\boldsymbol{g}}_k^{(5)}) = \mathrm{Spin}(7)$ and $(M_k, \boldsymbol{g}_k^{(5)})$ is a proper G_2 manifold.

We now proceed to a discussion of the metric singularities. The orbifold singularity of the base space $\mathcal{O}_k$ emerges at the boundaries where a certain component of the metric vanishes. To understand the effect of this singularity in the total space M_k, it is useful to see the behavior of the metric $\boldsymbol{g}_k^{(\ell)}$ with weak holonomy G_2 near boundaries. From (2.23) and (2.24), putting $\Omega(k) = k^2 + (k-2)^2$ we find

$$\begin{aligned}\boldsymbol{g}_k^{(\ell)} &\to \mathrm{d}t^2 + \frac{4t^2}{\Omega^2(k)}((k-2)\sigma_1 + k\hat{\sigma}_1)^2\\ &\quad + \frac{\ell\beta_\ell k}{k-2}(\sigma_2^2 + \sigma_3^2) + \beta_\ell(\hat{\sigma}_2^2 + \hat{\sigma}_3^2) + \frac{\beta_\ell}{(k-2)^2}(k\sigma_1 - (k-2)\hat{\sigma}_1)^2\end{aligned} \tag{3.33}$$

for $t \to t_1$, and

$$\begin{aligned}\boldsymbol{g}_k^{(\ell)} &\to \mathrm{d}t^2 + \frac{t^2}{25}(\sigma_3 + 3\hat{\sigma}_3)^2\\ &\quad + M^2(\sigma_1^2 + \sigma_2^2) + \beta_\ell(\hat{\sigma}_1^2 + \hat{\sigma}_2^2) + \beta_\ell(3\sigma_3 - \hat{\sigma}_3)^2\end{aligned} \tag{3.34}$$

for $t \to t_2$. These expressions correspond to the asymptotic forms (2.15) and (2.16) of the Tod–Hitchin metric. An important difference is that the collapsing circle is twisted by the fiber SO(3), which allows us to resolve the orbifold singularity of $\mathcal{O}_k$ as shown below. Let us represent the invariant 1-forms $\sigma_i, \hat{\sigma}_j$ in terms of Euler's angles:

$$\begin{aligned}&\sigma_1 = \mathrm{d}\psi + \cos\theta\mathrm{d}\phi, \quad \hat{\sigma}_1 = -\mathrm{d}\hat{\phi} - \cos\hat{\theta}\mathrm{d}\hat{\psi},\\ &\sigma_2 = \cos\psi\mathrm{d}\theta + \sin\psi\sin\theta\mathrm{d}\phi, \quad \hat{\sigma}_2 = -\cos\hat{\phi}\mathrm{d}\hat{\theta} - \sin\hat{\phi}\sin\hat{\theta}\mathrm{d}\hat{\psi},\\ &\sigma_3 = -\sin\psi\mathrm{d}\theta + \cos\psi\sin\theta\mathrm{d}\phi, \quad \hat{\sigma}_3 = -\sin\hat{\phi}\mathrm{d}\hat{\theta} + \cos\hat{\phi}\sin\hat{\theta}\mathrm{d}\hat{\psi}.\end{aligned} \tag{3.35}$$

The following transformation:

$$\eta = \frac{2}{\Omega(k)}((k-2)\psi - k\hat{\phi}), \quad \chi = k\psi + (k-2)\hat{\phi}, \tag{3.36}$$

yields

$$\begin{aligned} \boldsymbol{g}_k^{(\ell)} \to\ & \mathrm{d}t^2 + t^2\left(\mathrm{d}\eta + \frac{2(k-2)}{\Omega(k)}\cos\theta\mathrm{d}\phi - \frac{2k}{\Omega(k)}\cos\hat{\theta}\mathrm{d}\hat{\psi}\right)^2 \\ & + \frac{\ell\beta_\ell k}{k-2}(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2) + \beta_\ell(\mathrm{d}\hat{\theta}^2 + \sin^2\hat{\theta}\mathrm{d}\hat{\psi}^2) \\ & + \frac{\beta_\ell}{(k-2)^2}(\mathrm{d}\chi + k\cos\theta\mathrm{d}\phi + (k-2)\cos\hat{\theta}\mathrm{d}\hat{\psi})^2 \end{aligned} \tag{3.37}$$

for $t \to t_1$. From (3.36) we have $\mathrm{d}\eta \wedge \mathrm{d}\chi = 2(\mathrm{d}\psi \wedge \hat{\phi})$. It follows that one can adjust the ranges of the new angles as $0 \le \eta < 2\pi$, $0 \le \chi < 4\pi$ since Euler's angles have the ranges $0 \le \psi < 2\pi$, $0 \le \hat{\phi} < 2\pi$. Thus, the metric $\boldsymbol{g}_k^{(\ell)}$ extends smoothly over the circle bundle $T^{k,k-2}$ with the squashed metric

$$\begin{aligned} g_{\mathrm{Bolt}} =\ & \frac{\ell k}{k-2}(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2) + \mathrm{d}\hat{\theta}^2 + \sin^2\hat{\theta}\mathrm{d}\hat{\psi}^2 \\ & + \frac{1}{(k-2)^2}(\mathrm{d}\chi + k\cos\theta\mathrm{d}\phi + (k-2)\cos\hat{\theta}\mathrm{d}\hat{\psi})^2 \end{aligned} \tag{3.38}$$

at the boundary $t = t_1$. Also, similar arguments show that the metric extends over $T^{3,1}$ at $t = t_2$.

4. Geodesics on weak G_2 manifolds

In this section, we consider a Hamiltonian formulation describing geodesics on the weak G_2 manifold M_k. The phase space is the cotangent bundle T^*M_k with coordinates $(x^\alpha) = (t, \theta, \phi, \psi, \hat{\theta}, \hat{\phi}, \hat{\psi})$ and their conjugate momenta (p_α). The equations for geodesic flow are the canonical equations on T^*M_k with Hamiltonian $H = \frac{1}{2}g^{\alpha\beta}p_\alpha p_\beta$. Using the metric (3.31), we may write them explicitly as

$$\begin{aligned} H =\ & \frac{1}{2}p_t^2 + \frac{1}{2}\left(\frac{L_1^2}{a^2} + \frac{L_2^2}{b^2} + \frac{L_3^2}{c^2}\right) + \frac{1}{2\beta_\ell}(\hat{R}_1^2 + \hat{R}_2^2 + \hat{R}_3^2) \\ & + \frac{1}{2}\left(\frac{K_1^2\hat{R}_1^2}{a^2} + \frac{K_2^2\hat{R}_2^2}{b^2} + \frac{K_3^2\hat{R}_3^2}{c^2}\right) + \frac{K_1L_1\hat{R}_1}{a^2} + \frac{K_2L_2\hat{R}_2}{b^2} + \frac{K_3L_3\hat{R}_3}{c^2}. \end{aligned} \tag{4.39}$$

The functions L_i and $\hat{R}_j$ are canonically conjugate to σ_i and $\hat{\sigma}_j$, respectively:

$$\begin{aligned} L_1 &= p_\psi, \\ L_2 &= -\cot\theta\sin\psi p_\psi + \cos\psi p_\theta + \frac{\sin\psi}{\sin\theta}p_\phi, \\ L_3 &= -\cot\theta\cos\psi p_\psi - \sin\psi p_\theta + \frac{\cos\psi}{\sin\theta}p_\phi, \\ \hat{R}_1 &= -p_{\hat{\phi}}, \\ \hat{R}_2 &= \cot\hat{\theta}\sin\hat{\phi}p_{\hat{\phi}} - \cos\hat{\phi}p_{\hat{\theta}} - \frac{\sin\hat{\phi}}{\sin\theta}p_{\hat{\psi}}, \\ \hat{R}_3 &= -\cot\hat{\theta}\cos\hat{\phi}p_{\hat{\phi}} - \sin\hat{\phi}p_{\hat{\theta}} + \frac{\cos\hat{\phi}}{\sin\hat{\theta}}p_{\hat{\psi}}, \end{aligned} \tag{4.40}$$

which satisfy the SO(3) $\times$ SO(3) relations, $\{L_i, L_j\} = -\epsilon_{ijk}L_k$ and $\{\hat{R}_i, \hat{R}_j\} = -\epsilon_{ijk}\hat{R}_k$. We also introduce functions $\hat{L}_i$ and R_j by exchanging Euler's angles, $(\theta, \phi, \psi) \leftrightarrow (\hat{\theta}, \hat{\phi}, \hat{\psi})$. Then, one can easily show that they express the isometry SO(3) $\times$ SO(3) of the metric; $\{L_i, R_j\} = \{\hat{L}_i, \hat{R}_j\} = 0$ and hence $\{H, \hat{L}_i\} = \{H, R_j\} = 0$. It should be noticed that in general neither L_i nor $\hat{R}_j$ is conserved, although $\sum_i L_i^2 = \sum_i R_i^2$ and $\sum_i \hat{L}_i^2 = \sum_i \hat{R}_i^2$ are conserved quantities, the second Casimir. The relation between $L_i(\hat{L}_i)$ and $R_i(\hat{R}_i)$ corresponds to the relation between left and right actions of SO(3). The Hamiltonian equations $\frac{\mathrm{d}f}{\mathrm{d}\tau} = \{f, H\}$ are

$$\begin{aligned}
\frac{\mathrm{d}L_1}{\mathrm{d}\tau} &= \left(\frac{1}{c^2} - \frac{1}{b^2}\right) L_2L_3 - \frac{K_2}{b^2}L_3\hat{R}_2 + \frac{K_3}{c^2}L_2\hat{R}_3, \\
\frac{\mathrm{d}L_2}{\mathrm{d}\tau} &= \left(\frac{1}{a^2} - \frac{1}{c^2}\right) L_3L_1 - \frac{K_3}{c^2}L_1\hat{R}_3 + \frac{K_1}{a^2}L_3\hat{R}_1, \\
\frac{\mathrm{d}L_3}{\mathrm{d}\tau} &= \left(\frac{1}{b^2} - \frac{1}{a^2}\right) L_1L_2 - \frac{K_1}{a^2}L_2\hat{R}_1 + \frac{K_2}{b^2}L_1\hat{R}_2,
\end{aligned} \tag{4.41}$$

and

$$\begin{aligned}
\frac{\mathrm{d}\hat{R}_1}{\mathrm{d}\tau} &= \left(\left(\frac{K_3}{c}\right)^2 - \left(\frac{K_2}{b}\right)^2\right)\hat{R}_2\hat{R}_3 - \frac{K_2}{b^2}\hat{R}_3L_2 + \frac{K_3}{c^2}\hat{R}_2L_3, \\
\frac{\mathrm{d}\hat{R}_2}{\mathrm{d}\tau} &= \left(\left(\frac{K_1}{a}\right)^2 - \left(\frac{K_3}{c}\right)^2\right)\hat{R}_3\hat{R}_1 - \frac{K_3}{c^2}\hat{R}_1L_3 + \frac{K_1}{a^2}\hat{R}_3L_1, \\
\frac{\mathrm{d}\hat{R}_3}{\mathrm{d}\tau} &= \left(\left(\frac{K_2}{b}\right)^2 - \left(\frac{K_1}{a}\right)^2\right)\hat{R}_1\hat{R}_2 - \frac{K_1}{a^2}\hat{R}_2L_1 + \frac{K_2}{b^2}\hat{R}_1L_2
\end{aligned} \tag{4.42}$$

together with

$$\begin{aligned}
\frac{\mathrm{d}t}{\mathrm{d}\tau} &= p_t, \\
\frac{\mathrm{d}p_t}{\mathrm{d}\tau} &= \frac{\dot{a}}{a^3}L_1^2 + \frac{\dot{b}}{b^3}L_2^2 + \frac{\dot{c}}{c^3}L_3^2 \\
&\quad - \frac{K_1}{a}\left(\frac{\dot{K}_1}{a} - \frac{K_1\dot{a}}{a^2}\right)\hat{R}_1^2 - \frac{K_2}{b}\left(\frac{\dot{K}_2}{b} - \frac{K_2\dot{b}}{b^2}\right)\hat{R}_2^2 - \frac{K_3}{c}\left(\frac{\dot{K}_3}{c} - \frac{K_3\dot{c}}{c^2}\right)\hat{R}_3^2 \\
&\quad - \left(\frac{\dot{K}_1}{a^2} - 2\frac{K_1\dot{a}}{a^3}\right)L_1\hat{R}_1 - \left(\frac{\dot{K}_2}{b^2} - 2\frac{K_2\dot{b}}{b^3}\right)L_2\hat{R}_2 \\
&\quad - \left(\frac{\dot{K}_3}{c^2} - 2\frac{K_3\dot{c}}{c^3}\right)L_3\hat{R}_3.
\end{aligned} \tag{4.43}$$

This system may be regarded as an interacting rigid body system with angular momenta L_i and $\hat{R}_j$. The moments of inertia are given by $(I_i) = (a, b, c)$ and $(\hat{I}_i) = (a/K_1, b/K_2, c/K_3)$, which have a non-trivial time dependence through Eq. (4.43). When we put $K_i = 0$, the interaction between L_i and $\hat{R}_j$ vanishes. Thus, the angular momenta $\hat{R}_j$ are constants, and the remaining Eqs. (4.41) and (4.43) describe the geodesics on the Tod–Hitchin manifold [11,20,24].

As a special solution, consider the case $L_2 = \hat{R}_2 = 0$ in Eqs. (4.41)–(4.43). Then, the angular momenta (L_1, L_3) and $(\hat{R}_1, \hat{R}_3)$ are constants. If we can find a parameter t_0 such that

$a(t_0) = c(t_0)$, we have $\frac{\mathrm{d}L_2}{\mathrm{d}\tau} = \frac{\mathrm{d}\hat{R}_2}{\mathrm{d}\tau} = 0$ after setting

$$K_3(t_0)L_1\hat{R}_3 - K_1(t_0)L_3\hat{R}_1 = 0,$$
$$(K_1^2(t_0) - K_3^2(t_0))\hat{R}_3\hat{R}_1 - K_3(t_0)L_3\hat{R}_1 + K_1(t_0)L_1\hat{R}_3 = 0. \quad (4.44)$$

In fact, one can show that the parameter t_0 exists from the behavior of the Painlevé VI solution (see Fig. 2). Finally, the equation $p_t = 0$ requires a further constraint for the angular momenta:

$$\frac{\dot{a}}{a}L_1^2 + \frac{\dot{c}}{c}L_3^2 + K_1\left(\frac{a\Lambda}{3} + K_1\frac{\dot{a}}{a}\right)\hat{R}_1^2 + K_3\left(\frac{a\Lambda}{3} + K_3\frac{\dot{c}}{c}\right)\hat{R}_3^2$$
$$+ \left(\frac{a\Lambda}{3} + 2K_1\frac{\dot{a}}{a}\right)L_1\hat{R}_1 + \left(\frac{a\Lambda}{3} + 2K_3\frac{\dot{c}}{c}\right)L_3\hat{R}_3 = 0, \quad (4.45)$$

where we have used an identity $\dot{K}_1 = \dot{K}_3 = -a\Lambda/3$ at $a = c$. If we consider the case $\hat{R}_1 = \hat{R}_3 = 0$, Eq. (4.44) is automatically satisfied, and (4.45) yields $(L_1/L_3)^2 = -(\dot{c}/\dot{a})(t_0)$ [24]. As a result, we find a class of geodesics on M_k. For cases $k = 3, 4, 6$ and 8 given by (2.10)–(2.14), the solutions are summarized as follows:

(a) $k = 3$: $t_0 = \pi/6$
$\frac{L_1}{L_3} = \pm 1, \quad \hat{R}_1 = \hat{R}_3 = 0,$
$\frac{L_1}{\hat{R}_3} = \frac{\hat{R}_1}{\hat{R}_3} - \sqrt{3}$, $\frac{L_3}{\hat{R}_3} = 1 + \sqrt{3}$,
$\frac{L_1}{\hat{R}_1} = -2/(1 + \sqrt{3})$ and $-13/(3 + 4\sqrt{3})$, $L_3 = \hat{R}_3 = 0$.
(b) $k = 4$: $t_0 = \pi/6$
$\frac{L_1}{L_3} = \pm 2$, $\hat{R}_1 = \hat{R}_3 = 0$,
$\frac{L_1}{\hat{R}_3} = -\sqrt{3}\,\frac{\hat{R}_1}{\hat{R}_3}$, $\frac{L_3}{\hat{R}_3} = \sqrt{3}/2$,
$\frac{L_1}{\hat{R}_1} = \sqrt{3}$ and $-4\sqrt{3}/3$, $L_3 = \hat{R}_3 = 0$.
(c) $k = 6$: $r_0 = 2^{1/3} + 2^{-1/3} \cong 2.05$
$\frac{L_1}{L_3} \cong \pm 1.92$, $\hat{R}_1 = \hat{R}_3 = 0$,
$\frac{L_1}{\hat{R}_1} \cong -1.71$ and -1.28, $L_3 = \hat{R}_3 = 0$,
$\frac{L_3}{\hat{R}_3} \cong 0.95$ and 1.06, $L_1 = \hat{R}_1 = 0$.
(d) $k = 8$: $r_0 \cong 0.55$
$\frac{L_1}{L_3} \cong \pm 2.21$, $\hat{R}_1 = \hat{R}_3 = 0$,
$\frac{L_1}{\hat{R}_1} \cong -1.15$, $\frac{L_3}{\hat{R}_1} \cong \pm 0.50$, $\frac{\hat{R}_3}{\hat{R}_1} \cong \pm 0.52$,
$\frac{L_1}{\hat{R}_1} \cong -1.46$ and -1.15, $L_3 = \hat{R}_3 = 0$,
$\frac{L_3}{\hat{R}_3} \cong 0.97$ and 1.03, $L_1 = \hat{R}_1 = 0$.

5. M-theory on $\mathrm{AdS}_4 \times M_k$

We have constructed infinitely many compact Einstein manifolds M_k, which are 3-Sasakian manifolds for $\ell = 1$ and proper weak G_2 manifolds for $\ell = 5$. The orbifold singularity of the Tod–Hitchin geometry has been resolved by having additional dimensions, so we can expect the resolution of the orbifold singularity in the moduli by adding scalars in the corresponding gauge theory. The resulting seven-dimensional manifolds M_k admit 3-Sasakian or proper weak G_2 structures, and thus the gauge theories are $\mathcal{N} = 3$ supersymmetric for $\ell = 1$, while they are

$\mathcal{N} = 1$ supersymmetric for $\ell = 5$. It was shown that the manifold $M_3(\ell = 1) = N^{0,1,0}$ appears as the moduli space of an $\mathcal{N} = 3$ gauge theory [25]. We expect the seven-dimensional manifolds M_k with general k to also emerge as the moduli spaces of three-dimensional $\mathcal{N} = 3$ or $\mathcal{N} = 1$ supersymmetric gauge theories. It is interesting to achieve this and to reveal the role of k from the viewpoint of gauge theories. Leaving this interesting issue as a future problem, in this section we consider M-theory on $\mathrm{AdS}_4 \times M_k$, and apply the AdS/CFT correspondence.

Using the 3-Sasakian or proper weak G_2 manifolds M_k, one can construct supersymmetric M-theory solutions, $\mathrm{AdS}_4 \times M_k$, which are AdS/CFT dual to three-dimensional superconformal field theories. The isometry of M_k corresponds to the global symmetry of the dual superconformal field theories, including the R-symmetry. The manifolds M_k contain S^7, $N^{0,1,0}$ and squashed S^7 ($\tilde{S}^7$) as special homogeneous cases: $M_3(\ell = 1)$, $M_4(\ell = 1)$ and $M_3(\ell = 5)$, respectively. For these cases, the dual three-dimensional gauge theories which flow to the superconformal field theories at the IR are the $\mathcal{N} = 8$ gauge theory without flavor [2] for S^7 with SO(8) isometry, the $\mathcal{N} = 3$ gauge theory with SU(3) flavor [25,26] for $N^{0,1,0}$ with $\mathrm{SU}(3) \times \mathrm{SU}(2)$ isometry. The squashed S^7 admits $\mathrm{SO}(5) \times \mathrm{SO}(3)$ isometry, so the dual theory is expected to be $\mathcal{N} = 1$ gauge theory with $\mathrm{SO}(5) \times \mathrm{SO}(3)$ flavor. For generic k, because the metrics on M_k admit $\mathrm{SO}(3) \times \mathrm{SO}(3)$ isometry as shown in Section 4, the gauge theories which flow to the superconformal field theories at the IR are an $\mathcal{N} = 3$ gauge theory with SO(3) flavors for $\ell = 1$, and an $\mathcal{N} = 1$ gauge theory with $\mathrm{SO}(3) \times \mathrm{SO}(3)$ flavors for $\ell = 5$. Since it is not easy to extract the Kaluza–Klein spectrum on M_k as is expected from the analysis in Section 4, we assume this correspondence here. The UV limit of the theory is described by $\mathbb{R}^{1,2} \times C(M_k)$, where $C(M_k)$ stands for the cone over M_k. The cone metrics are hyperkähler for $\ell = 1$ and Spin(7) for $\ell = 5$. For the homogeneous cases S^7, $N^{0,1,0}$ and $\tilde{S}^7$, the holographic RG flows which interpolate $\mathbb{R}^{1,2} \times C(M_k)$ at UV and $\mathrm{AdS}_4 \times M_k$ at IR are examined in [27]. For general k, the brane solution which describes the holographic RG flow from $\mathbb{R}^{1,2} \times C(M_k)$ at UV to $\mathrm{AdS}_4 \times M_k$ at IR is

$$g_{11} = H^{-\frac{2}{3}} g_{\mathbb{R}^{1,2}} + H^{\frac{1}{3}} \bar{\boldsymbol{g}}_k^{(\ell)}, \quad F = \mathrm{dvol}(\mathbb{R}^{1,2}) \wedge \mathrm{d}H^{-1}, \quad H = 1 + \left(\frac{a}{r}\right)^6, \tag{5.46}$$

where $a = (2^5 \pi^2 N)^{\frac{1}{6}} \ell_P$ and $\bar{\boldsymbol{g}}_k^{(\ell)} = \mathrm{d}r^2 + r^2 \boldsymbol{g}_k^{(\ell)}$. This corresponds to N coincident M2-branes at $r = 0$. For small r, the brane solution (5.46) reduces to the product metric of M_k with cosmological constant $1/a^2$ and AdS_4 with $4/a^2$, and the 4-form strength $F = 6\mathrm{dvol}(\mathrm{AdS}_4)/a$. On the other hand, for large r, (5.46) approaches the product metric of $\mathbb{R}^{1,2}$ and $C(M_k)$ without the 4-form strength. It is interesting to examine the limit, $k \to \infty$ together with $\Lambda \to 0$, in which the four-dimensional base space, Tod–Hitchin geometry, converges to the Atiyah–Hitchin hyperkähler manifold M_{AH}. The limit $\Lambda \to 0$ corresponds to the limit $a \to \infty$, because the cosmological constant $\lambda = \Lambda \frac{2\ell - 1}{2\ell}$ of M_k is now $1/a^2$. In this limit, (5.46) approaches the metric on $\mathbb{R}^{1,3} \times \mathbb{R}^3/\mathbb{Z}_2 \times M_{AH}$ without the 4-form strength because M_k reduces to $\mathbb{R}^3/\mathbb{Z}_2 \times M_{AH}$. Apart from the $\mathbb{Z}_2$ factor, this solution can be regarded as an orientifold 6-plane of the IIA superstring theory [28], and thus the g_{11} provides an approximation of the orientifold plane.

Infinitely many inhomogeneous Einstein metrics on compact manifolds are derived from Kerr–de Sitter black holes as the Page limit in [29–31], and those with a Sasaki structure found in [32] as the Sasaki–Einstein twist in [33]. It is interesting to consider the black hole solutions corresponding to M_k constructed in this paper. We have discussed the holographic RG flow from $\mathbb{R}^{1,2} \times C(M_k)$ to $\mathrm{AdS}_4 \times M_k$. In [34], a transition from $\mathrm{AdS}_4 \times \tilde{S}^7$ to $\mathrm{AdS}_4 \times S^7$ is discussed. A similar transition from $\mathrm{AdS}_4 \times M_k(\ell = 5)$ to $\mathrm{AdS}_4 \times M_k(\ell = 1)$ is expected. We leave these issues for future investigations.

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Appendix A. Four-dimensional ASD Einstein manifolds

The Bianchi IX metric is of the form

$$g = \mathrm{d}t^2 + a^2(t)\sigma_1^2 + b^2(t)\sigma_2^2 + c^2(t)\sigma_3^2, \tag{A.1}$$

where σ_i are left-invariant 1-forms on SO(3),

$$\mathrm{d}\sigma_i = -\frac{1}{2}\epsilon_{ijk}\sigma_j \wedge \sigma_k. \tag{A.2}$$

Defining the vielbein

$$e^0 = \mathrm{d}t, \quad e^1 = a\sigma_1, \quad e^2 = b\sigma_2, \quad e^3 = c\sigma_3, \tag{A.3}$$

one evaluates the spin connection as

$$\begin{aligned}
\omega_{01} &= -\frac{\dot a}{a}e^1, \quad \omega_{12} = -\frac{a^2+b^2-c^2}{2abc}e^3,\\
\omega_{02} &= -\frac{\dot b}{b}e^2, \quad \omega_{31} = -\frac{a^2-b^2+c^2}{2abc}e^2,\\
\omega_{03} &= -\frac{\dot c}{c}e^3, \quad \omega_{23} = -\frac{-a^2+b^2+c^2}{2abc}e^1.
\end{aligned} \tag{A.4}$$

The Einstein equations $R_{\alpha\beta} = \Lambda\delta_{\alpha\beta}$ are given by

$$\begin{aligned}
&\frac{\ddot a}{a} + \frac{\ddot b}{b} + \frac{\ddot c}{c} + \Lambda = 0,\\
&\frac{\ddot a}{a} + \frac{\dot a}{a}\left(\frac{\dot b}{b} + \frac{\dot c}{c}\right) - \frac{a^4-(b^2-c^2)^2}{2a^2b^2c^2} + \Lambda = 0,\\
&\frac{\ddot b}{b} + \frac{\dot b}{b}\left(\frac{\dot a}{a} + \frac{\dot c}{c}\right) - \frac{b^4-(a^2-c^2)^2}{2a^2b^2c^2} + \Lambda = 0,\\
&\frac{\ddot c}{c} + \frac{\dot c}{c}\left(\frac{\dot a}{a} + \frac{\dot b}{b}\right) - \frac{c^4-(a^2-b^2)^2}{2a^2b^2c^2} + \Lambda = 0.
\end{aligned} \tag{A.5}$$

The ASD condition further requires the following equations:

$$\begin{aligned}
&\frac{\ddot a}{a} + \left(C\frac{\dot b}{b} + B\frac{\dot c}{c} - \frac{\dot a}{bc}\right) + \frac{\Lambda}{3} = 0,\\
&\frac{\ddot b}{b} + \left(C\frac{\dot a}{a} + A\frac{\dot c}{c} - \frac{\dot b}{ac}\right) + \frac{\Lambda}{3} = 0,
\end{aligned}$$

$$\frac{\ddot{c}}{c}+\left(B\frac{\dot{a}}{a}+A\frac{\dot{b}}{b}-\frac{\dot{c}}{ab}\right)+\frac{\Lambda}{3}=0,$$
$$\frac{\dot{a}\dot{b}}{ab}-\frac{a^4+b^4-3c^4+2(-a^2b^2+b^2c^2+a^2c^2)}{4a^2b^2c^2}+\left(B\frac{\dot{a}}{a}+A\frac{\dot{b}}{b}-\frac{\dot{c}}{ab}\right)+\frac{\Lambda}{3}=0,$$
$$\frac{\dot{a}\dot{c}}{ac}-\frac{a^4-3b^4+c^4+2(a^2b^2+b^2c^2-a^2c^2)}{4a^2b^2c^2}+\left(C\frac{\dot{a}}{a}+A\frac{\dot{c}}{c}-\frac{\dot{b}}{ac}\right)+\frac{\Lambda}{3}=0,$$
$$\frac{\dot{b}\dot{c}}{bc}-\frac{-3a^4+b^4+c^4+2(a^2b^2-b^2c^2+a^2c^2)}{4a^2b^2c^2}+\left(C\frac{\dot{b}}{b}+B\frac{\dot{c}}{c}-\frac{\dot{a}}{bc}\right)+\frac{\Lambda}{3}=0, \quad (A.6)$$

where

$$A=\frac{-a^2+b^2+c^2}{2abc},\quad B=\frac{a^2-b^2+c^2}{2abc},\quad C=\frac{a^2+b^2-c^2}{2abc}. \quad (A.7)$$

Appendix B. Tod–Hitchin metric

Tod [9] and Hitchin [10,11] studied the Bianchi IX metric written in the form

$$g^{\mathrm{TH}}=H(x)\left(\frac{\mathrm{d}x^2}{x(1-x)}+\frac{\sigma_1^2}{\Omega_1(x)^2}+\frac{(1-x)\sigma_2^2}{\Omega_2(x)^2}+\frac{x\sigma_3^2}{\Omega_3(x)^2}\right). \quad (B.8)$$

They showed that g^{TH} gives an ASD Einstein metric with positive cosmological constant if the functions Ω_i satisfy a set of first-order equations

$$\Omega_1'=-\frac{\Omega_2\Omega_3}{x(1-x)},\quad \Omega_2'=-\frac{\Omega_3\Omega_1}{x},\quad \Omega_3'=-\frac{\Omega_1\Omega_2}{1-x}, \quad (B.9)$$

where a prime denotes a derivative with respect to x, and the conformal factor H is given by

$$H=-\frac{8x\Omega_1^2\Omega_2^2\Omega_3^2+2\Omega_1\Omega_2\Omega_3\left\{x(\Omega_1^2+\Omega_2^2)-(1-4\Omega_3^2)(\Omega_2^2-(1-x)\Omega_1^2)\right\}}{4\left\{x\Omega_1\Omega_2+2\Omega_3\left(\Omega_2^2-(1-x)\Omega_1^2\right)\right\}^2}. \quad (B.10)$$

Writing the functions Ω_i^2 in terms of $y(x)$ as

$$\Omega_1^2=\frac{(y-x)^2y(y-1)}{x(1-x)}\left(z-\frac{1}{2(y-1)}\right)\left(z-\frac{1}{2y}\right),$$
$$\Omega_2^2=\frac{y^2(y-1)(y-x)}{x}\left(z-\frac{1}{2(y-x)}\right)\left(z-\frac{1}{2(y-1)}\right),$$
$$\Omega_3^2=\frac{(y-1)^2y(y-x)}{(1-x)}\left(z-\frac{1}{2y}\right)\left(z-\frac{1}{2(y-x)}\right), \quad (B.11)$$

together with an auxiliary variable

$$z=\frac{x-2xy+y^2-2x(1-x)y'}{4y(y-1)(y-x)}, \quad (B.12)$$

one can reduce the first-order equations (B.9) to a single second-order differential equation, i.e. Painlevé VI equation:

$$\begin{aligned} y'' &= \frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-x}\right)y'^2-\left(\frac{1}{x}+\frac{1}{x-1}+\frac{1}{y-x}\right)y' \\ &\quad +\frac{y(y-1)(y-x)}{x^2(x-1)^2}\left(\alpha+\beta\frac{x}{y^2}+\gamma\frac{x-1}{(y-1)^2}+\delta\frac{x(x-1)}{(y-x)^2}\right), \end{aligned} \tag{B.13}$$

with $(\alpha,\beta,\gamma,\delta)=(1/8,-1/8,1/8,\ 3/8)$.

Appendix C. G_2-structure

We assume the diagonal form of the Kaluza–Klein metric (3.27),

$$g_{\rm diag}=\mathrm{d}t^2+a^2(t)\sigma_1^2+b^2(t)\sigma_2^2+b^2(t)\sigma_3^2+\alpha_1^2(\phi^1)^2+\alpha_2^2(\phi^2)^2+\alpha_3^2(\phi^3)^2. \tag{C.14}$$

Provided the self-dual instanton $\phi^i=s_{ji}A^j+\tilde{\sigma}_i$, the curvature $\Theta^i=\mathrm{d}\phi^i+\frac{1}{2}\epsilon_{ijk}\phi^j\wedge\phi^k$ is calculated as

$$\Theta^i=-\frac{\Lambda}{3}s_{ji}\left(e^0\wedge e^j+\frac{1}{2}\epsilon_{jk\ell}e^k\wedge e^\ell\right), \tag{C.15}$$

where $(s_{ij})\in \mathrm{SO}(3)$ and $\{e^\mu;\mu=0,1,2,3\}$ is the orthonormal basis of the Bianchi IX metric defined by (A.3). We now introduce an orthonormal basis of the Kaluza–Klein metric: $\theta^i=\alpha_i\phi^i$ $(i=1,2,3)$ for the fiber metric, and θ^α $(\alpha=4,5,6,7)$ are defined by the following equations:

$$\begin{aligned} &\Theta^1=\frac{\Lambda}{3}(\theta^4\wedge\theta^5+\theta^6\wedge\theta^7),\qquad \Theta^2=\frac{\Lambda}{3}(\theta^4\wedge\theta^6+\theta^7\wedge\theta^5),\\ &\Theta^3=\frac{\Lambda}{3}(\theta^4\wedge\theta^7+\theta^5\wedge\theta^6) \end{aligned} \tag{C.16}$$

and (C.15). Then, the 3-form (3.32) can be written as

$$\omega=\alpha_1\alpha_2\alpha_3\phi^1\wedge\phi^2\wedge\phi^3+\frac{3}{\Lambda}(\alpha_1\phi^1\wedge\Theta^1+\alpha_2\phi^2\wedge\Theta^2+\alpha_3\phi^3\wedge\Theta^3). \tag{C.17}$$

Thus, the G_2-equation $\mathrm{d}\omega=c*\omega$ reduces to the algebraic equations

$$\begin{aligned} &\alpha_1+\alpha_2+\alpha_3=\frac{3c}{2\Lambda}\\ &\alpha_1\alpha_2\alpha_3+\frac{3}{\Lambda}(-\alpha_1+\alpha_2+\alpha_3)=\frac{3c}{\Lambda}\alpha_2\alpha_3, \end{aligned} \tag{C.18}$$

and the two equations obtained by cyclically permuting $\alpha_1,\alpha_2,\alpha_3$. These reproduce the solution (3.30) and hence the metric (3.31).

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